

Statistical analysis of Leonardo's rule



Leonardo da Vinci observed in his notebooks that “all the branches of a tree at every stage of its height when put together are equal in thickness to the trunk” [1], which means that when a mother branch of diameter d splits into N daughter branches of diameters d_i , the following relation holds on average

$$d^\Delta = \sum_{i=1}^N d_i^\Delta$$

where the Leonardo exponent is ≈ 2 . Surprisingly, there have been few assessments of this rule.

In [2] it is proposed that Leonardo's rule is a consequence of the self-similarity of the tree trunk and wind-induced stress. In the mentioned paper some ad hoc hypothesis, based in fractal theory and fracture theory, were introduced in order to obtain Leonardo's law.

However, it is curious that Leonardo himself proposes a very simple explanation of this rule based on the characteristics of uid motion. \When a branch grows, Leonardo argues, its thickness will depend on the amount of sap it receives from the one below the branching point. In the tree as a whole, there is a constant flow of sap, which rises up through the trunk and divides between the branches owing through successive ramifications.

the total quantity of sap carried by the tree is constant, the quantity carried by each branch will be proportional to its cross section, so the total cross section at each level will be equal to that of a trunk" [3].

This argument is naive, since the flux of the sap, as viscous liquid, is not proportional to the cross section of the branch, but to the fourth power of its radius, if we consider branches and trunk as cylinders.

In our opinion, though Leonardo's rule is not a trivial result from fluid mechanics, a simple and direct approach, based on mass conservation, can be made to explain this observation.

The amount of fluid that goes through a conduit of this type is given by the Poiseuille's Law

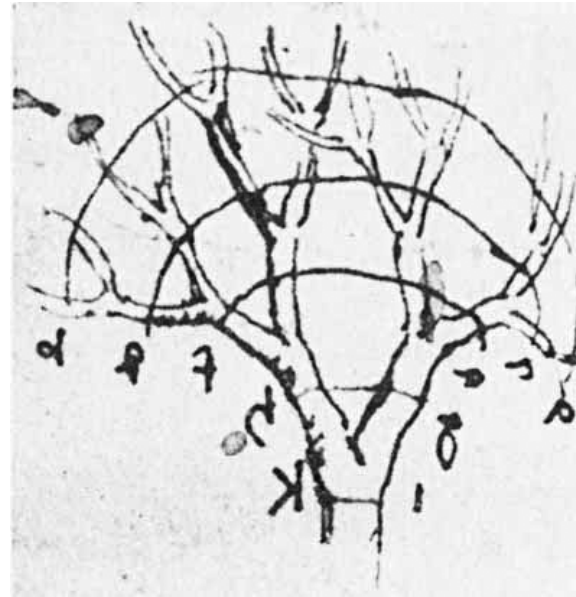
$$Q(R) = \frac{\pi \nabla P}{8\eta} R^4$$

Q is the flux

Da Vinci: Codex Atlanticus

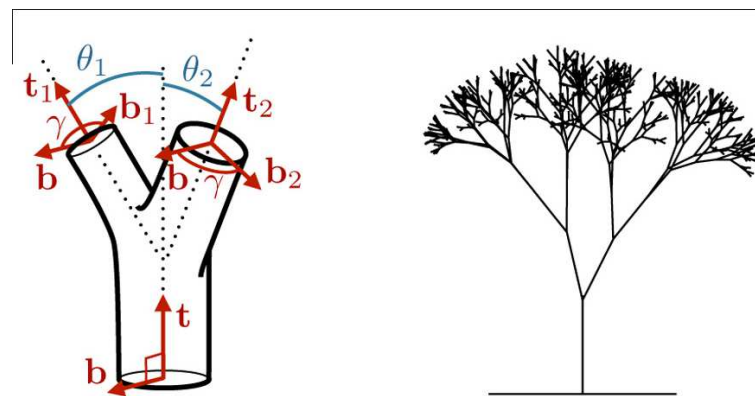


$$r_1^2 + r_2^2 = r^2$$



C. Eloy, Phys. Rev. Lett, 107, 258101, (2011)

Criterion: resistance to fracture



Let's apply the criterion of Leonardo, but considering the viscosity of sap and that the flux goes through a distribution of vessels which obey a Lévi distribution of radius. The origin of this distribution will be justified in the next pages.

If we calculate the total flux Q on a given branch considering that it is distributed among a network of "pipes" Levi-distributed by radius, the total flux of sap transported by the branch is the sum of all contributions of the vessels.

$$Q(R) = \int_0^R f(r)Q(r)dr$$

$$Q(r) \sim r^4$$

$$f(r) \sim r^{-x}$$

(asymptotic behavior)

Then, to fulfill Leonardo's rule x must be:

$$x = 3$$

How to justify this?



Papers of Murray and Christophe

letters to nature

every branch rank. Instead, the optimum network has a finite number of wide conduits at the base that find an increasing number of narrower conduits distally. This follows from the application of Murray's law, which predicts the optimal taper of blood vessels in the cardiovascular system. Our measurements of plant xylem indicate that these conduits conform to the Murray's law optimum as long as they do not function additionally as supports for the plant body.

The idea that organisms may be optimally designed has fueled considerable research and controversy¹⁻⁴. Both animals and plants require extensive and expensive transport systems, for which an optimal design presumably provides a selective advantage. In plants, transport through xylem has allowed growth in height and colonization of diverse habitats⁵, and must be extensive because photosynthesis entails an exchange of large quantities of inert water for atmospheric carbon dioxide. Plants use capillary action to transport water from the roots to the leaves without using metabolic energy, and the xylem conduits themselves are cheap to maintain, being dead at maturity. However, the numerous thick-walled conduits require considerable carbon investment. This cost, combined with the benefit to photosynthesis of optimizing xylem design, implies that it would be advantageous for plants to maximize hydraulic conductance per vascular investment, and hence, to obey Murray's law¹.

In 1926, Murray¹ developed a theory of optimal cardiovascular design that serves for the flow of blood through the aorta through progressive branch points to the capillaries that maximize the hydraulic conductance of flow through the vascular network for a fixed investment in blood and vessel volume. Under ideal conditions, the optimal design equalizes the sum of all radii cubed ($\sum R^3$) at all points along the flow path if the volume flow rate (Q) of the blood is conserved within the vascular system and the flow is laminar¹. This result, known as Murray's law, fits data better than equalizing the sum of radii raised to the second or fourth power^{1,11}. Surprisingly, this law has not been systematically applied and tested in plant vascular¹².

Murray's law is applicable to plant xylem given the following four conditions: (1) The steady-state xylem flow is constant along the flow path. With the exception of the absorbing roots and the minor leaf veins, there will be no net loss of water from the xylem to surrounding tissues under steady-state conditions. (2) Xylem hydraulic conductance is proportional to conduit radius raised to the fourth power. This is the Hagen-Poiseuille prediction for laminar flow through cylindrical capillaries, which are appropriate

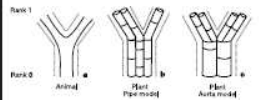


Figure 1 Transport networks. (a) A simple network consisting of a central stem and two daughter branches. (b) A network that conserves the sum of radii cubed ($\sum R^3$) at all points along the flow path. (c) A network that conserves the sum of radii to the fourth power ($\sum R^4$) at all points along the flow path. The optimal design equalizes the sum of all radii cubed ($\sum R^3$) at all points along the flow path if the volume flow rate (Q) of the blood is conserved within the vascular system and the flow is laminar¹. This result, known as Murray's law, fits data better than equalizing the sum of radii raised to the second or fourth power^{1,11}. Surprisingly, this law has not been systematically applied and tested in plant vascular¹².

approximations for low-velocity xylem flow. (3) Conduit wall volume is proportional to total conduit volume. Murray assumed that the major volume cost was in the blood, and that the thin walls of blood vessels were negligible. In plants it is the opposite: the xylem walls are cheap but the conduit walls are costly. Murray's law is still valid as long as the conduit wall volume scales directly with total conduit volume. The wall volume:conduit volume is proportional to the curvature resistance of xylem¹³, so it is necessary to withstand the compressive forces on the wall caused by negative pressure. Thus, for a given curvature resistance throughout the system (wood), Murray's law holds. (4) The xylem conduits must function primarily in transport, as opposed to providing an additional mechanical support role. When conduits are serving both functions (as in conifer shoots or diffuse-porous trees), the optimization criteria must include mechanical as well as hydraulic considerations, and Murray's law is inappropriate.

Murray's law by itself is not sufficient to characterize the fundamental versus investment trade-off in xylem, because of dimensional structural differences between the vasculature of animals and plants. Figure 1 illustrates the angle-branched case of the animal network. As the single parent tube at rank 0 (for example, the

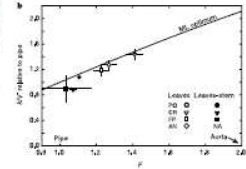
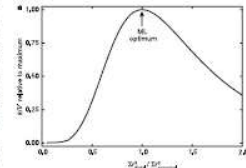


Figure 2 Murray's law. (a) The Murray's law (ML) optimum. The best possible hydraulic conductance ($\sum R^3$) per fixed volume ($\sum R^4$) is obtained by equalizing the sum of radii cubed ($\sum R^3$) at all points along the flow path in a branched vascular network. The optimum $\sum R^3$ occurs at a $\sum R^4$ value¹ according to Murray's law regardless of the conduit branching number (N). The Murray's law optimum occurs if the optimum is set to 1 for $N = 2$ pipe model. The greater efficiency of wide, central conduits to plants with higher fractal vascular branching resistance (a given volume over the pipe model) is predicted when the measured fractal tree is compared with Murray's law optimum with two open hydraulic units and compared with Murray's law optimum with one closed hydraulic unit. (b) Performance comparison of the ML optimum (black circles) with the $\sum R^4$ optimum (open circles) and the $\sum R^2$ optimum (open squares). The $\sum R^4$ optimum (black circles) is the best possible hydraulic conductance per fixed volume.

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JPL Leonardo Was Wise

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Leonardo Was Wise Trees Conserve Cross-Sectional Area Despite Vessel Structure

Rizwan Aratsu

Abstract

Beginning with Leonardo da Vinci's assertion that trees conserve total cross-sectional area across every branching point, I tested ten species of trees in the vicinity of Princeton, New Jersey, to see if they do indeed adhere to the rule of conservation as asserted by the Italian master and those who followed him. Based on my review of the literature, I expected to find that trees would either conserve area or not depending on the porosity of their wood to water. To my surprise, I found that all ten species conserve cross-sectional area in approximately the same way despite large differences in porosity. In particular, their twigs roughly doubled in cross-sectional area across each branching while their larger branches approximated area-preservation, as Leonardo had predicted. Rather than precisely preserving area, the trees actually tended to increase in area ever so slightly as I moved from trunk to twig tips. For this reason, I describe a conical model of tree form originated by Horn (1998, in press), which may estimate the volume of a tree more accurately than the traditional cylindrical model.

Introduction

Leonardo's Place in History

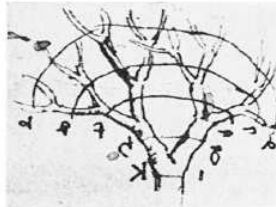


Figure 3. In his notebook, Leonardo da Vinci made this sketch depicting the branching pattern of trees. He deduced that the total thickness of branches along each of the axis would equal the thickness of the trunk. (Richter 1929, plate 27)

Many observers of nature—some scientists, some poets, some both—have attempted to explain the complex structure of trees. One of the most perceptive of these descriptions was made in the 15th

http://jyoi.zpl.org/submit/volume1/issue1/articles/aratsu.html

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Leonardo's Rule, Self-Similarity, and Wind-Induced Stresses in Trees

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Examining botanical trees, Leonardo da Vinci noted that the total cross section of branches is conserved across branching nodes. In this Letter, it is proposed that this rule is a consequence of the tree skeleton having a self-similar structure and the branch diameters being adjusted to resist wind-induced loads.

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Leonardo da Vinci observed in his notebooks that "all the branches of a tree at every stage of its height when put together are equal in thickness to the trunk" [1], which means that when a smaller branch of diameter d splits into N daughter branches of diameters d_i , the following relation holds on average

$$d^4 = \sum_{i=1}^N d_i^4 \quad (1)$$

where the Leonardo exponent is $\Delta = 2$. Surprisingly, there have been few assessments of this rule, but the available data indicate that the Leonardo exponent is in the interval $1.8 < \Delta < 2.3$ for a large number of species [2-4]. In fact, Leonardo's rule is so natural to the eye that it is routinely used in computer-generated trees [5]. Yet, alternative analyses of the branching geometry have been proposed based on analogies with river networks, bronchial tree, and arterial tree [6].

Two different models have been proposed to explain Leonardo's rule: the pipe model [7], which assumes that trees are a collection of identical vascular vessels connecting the leaves to the roots, and the principle of elastic similarity [8,9], which postulates that the deflection of branches under their weight is proportional to their length. However, none of these explanations are convincing. The first because the portion of a branch cross section devoted to vascular transport (i.e., the sapwood) may be as low as 5% in mature trees and it seems thus dubious that the whole tree architecture is governed by hydraulic constraints. The second because the postulate behind elastic similarity is artificial, hard to relate to any adaptive advantage, and, furthermore, it seems unlikely that trees can respond to branch deflections.

In this Letter, an alternative explanation is offered: Leonardo's rule is a consequence of trees being designed to resist wind-induced stresses. Plants are known to respond to dynamic loading for a long time, a phenomenon called thigmomorphogenesis [10,11]. In that line of thinking, Metzger [12] proposed in the 19th century the constant stress model. This model states that the trunk diameter varies such that the bending stress due to wind

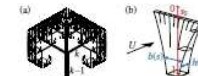
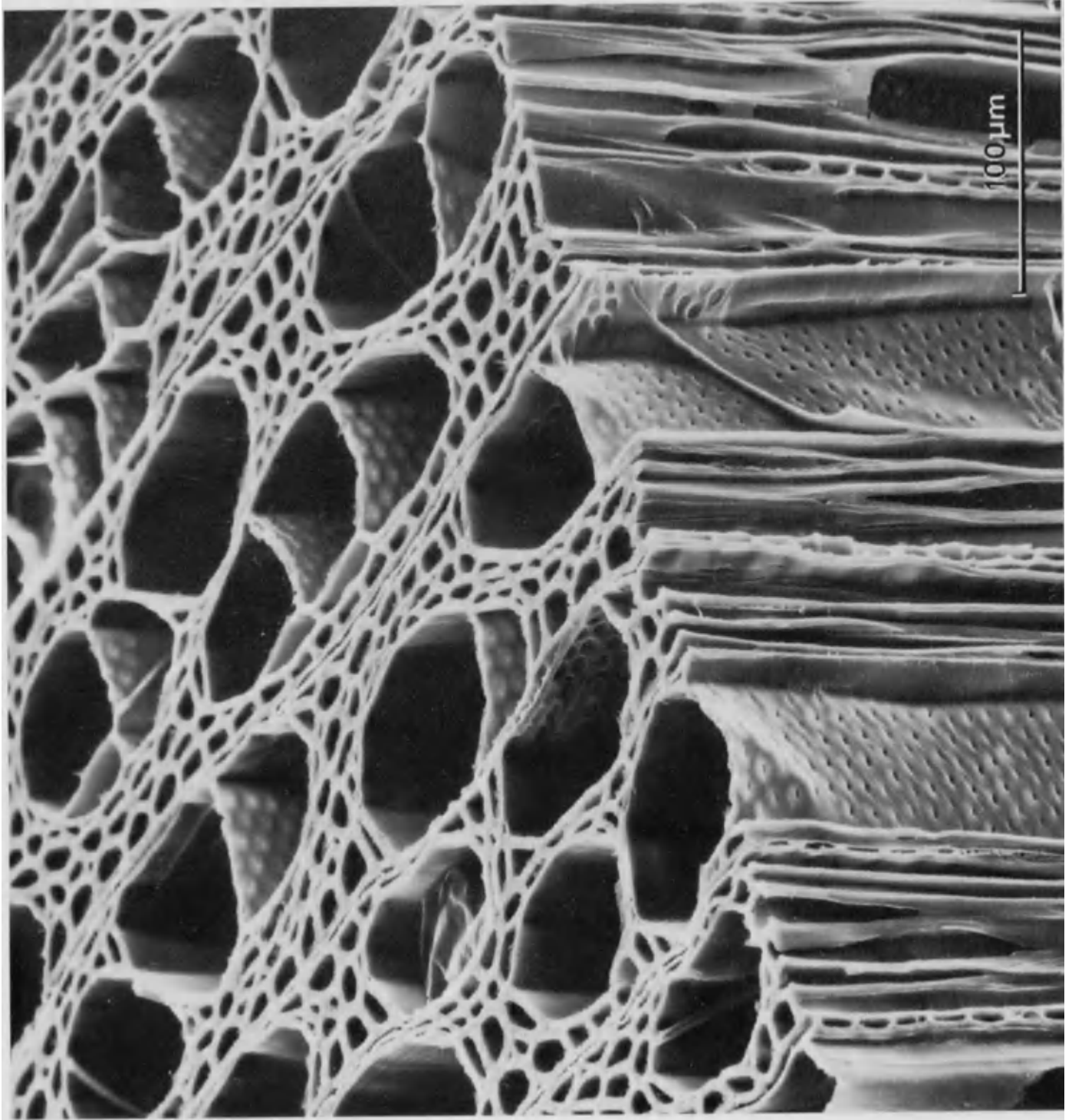


FIG. 4 (color online). Two analytical models: (a) the fractal tree model; (b) the continuous tapered beam model [8].

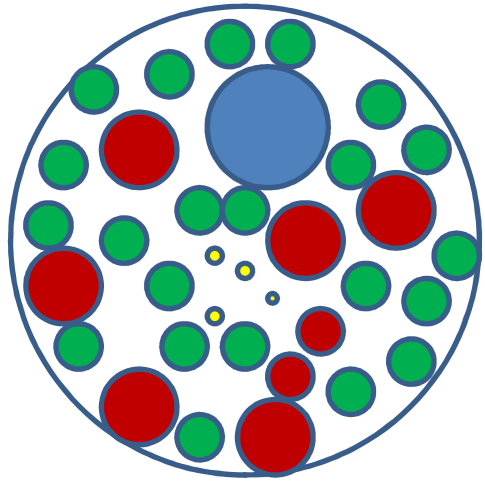
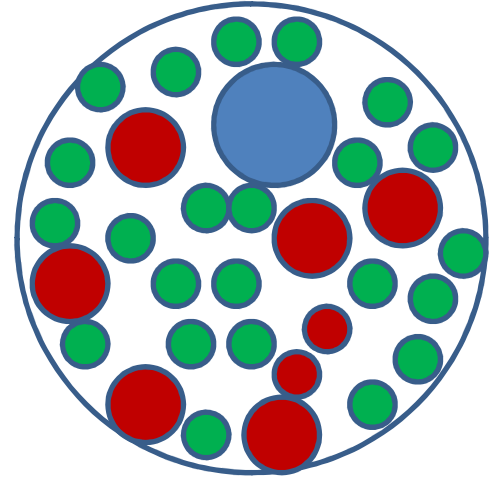
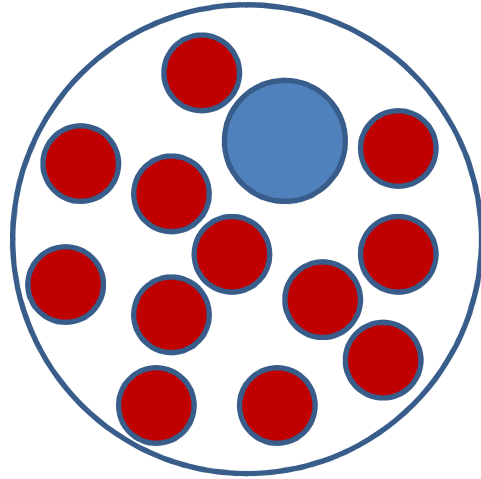
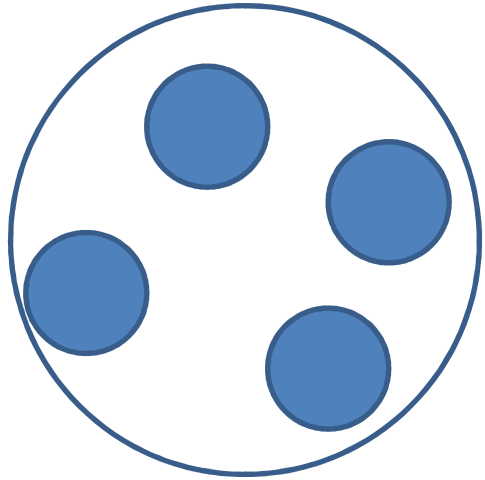
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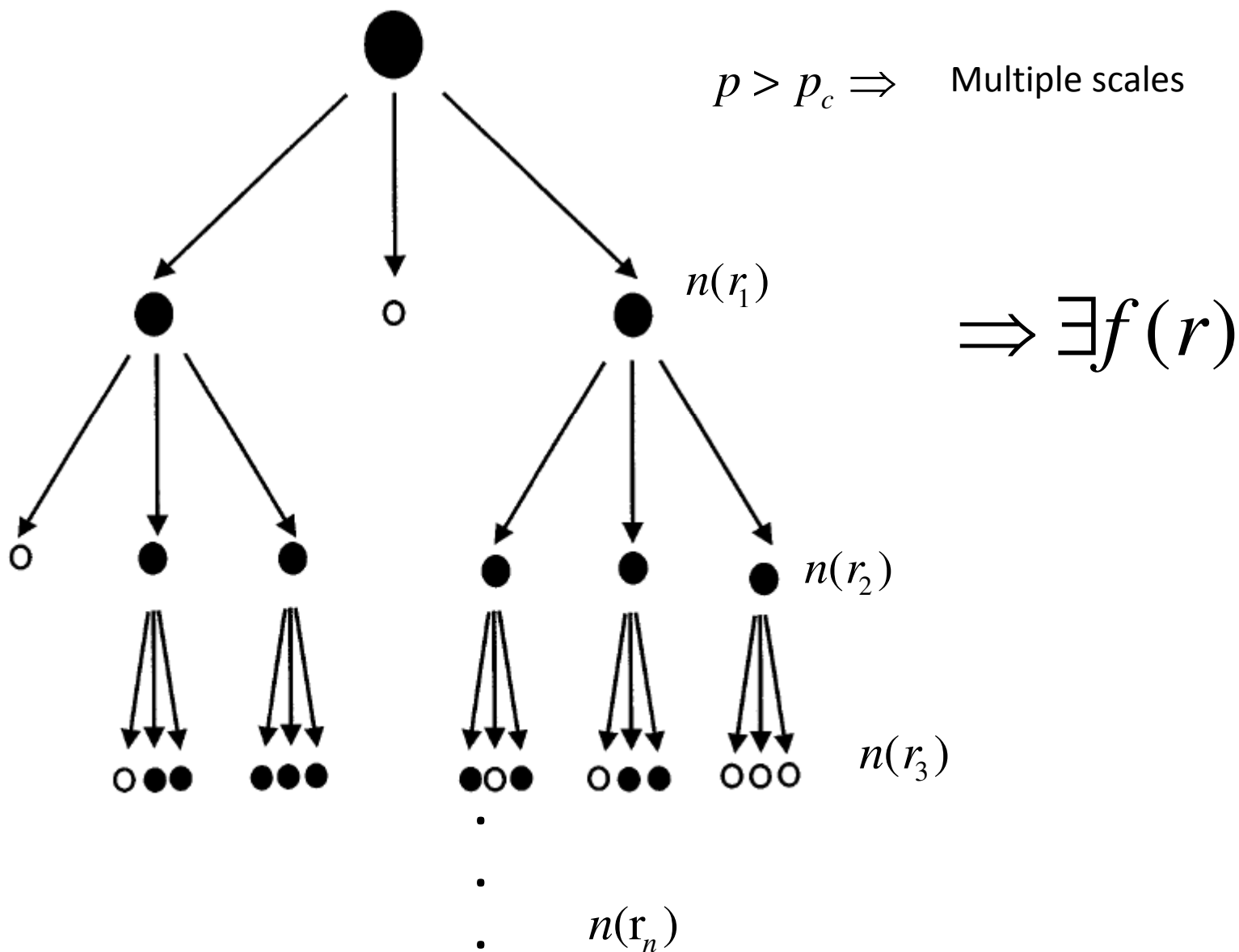
The previous photo of the wall between two vessels suggests a formation mechanism of evolutive type, i.e., a few big vessels bifurcate in smaller ones with a given probability, Then the new generation repeats the process, and so on, giving as a result a distribution of vessels broadly distributed, schematically illustrated as follows:



etc

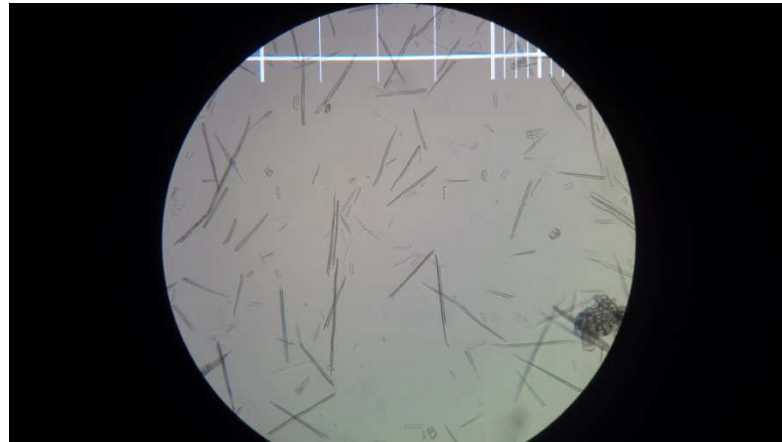
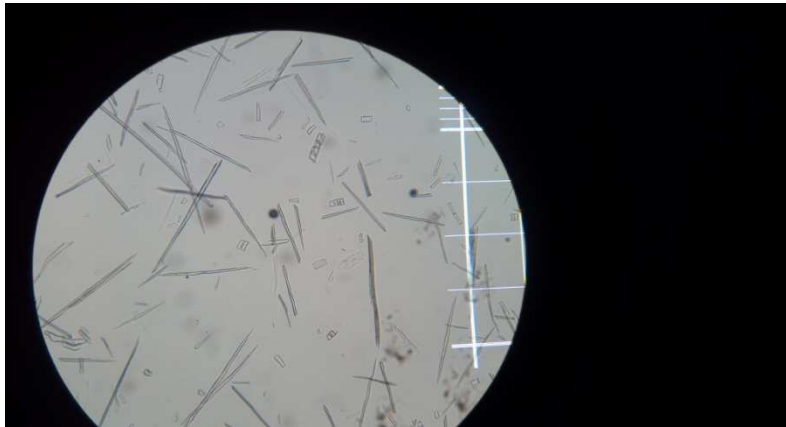
This process suggests a representation in terms of a Bethe lattice, where old vessels generate new ones with a given probability, thus disappearing, so that the problem becomes that of the percolation on a Bethe lattice. This is a well known problem, where the critical probability to percolate an infinite branch is known. In our case, let us consider that branching occurs conserving the flux, i.e., The new vessels conduct the same flux as the old one, progenitor.

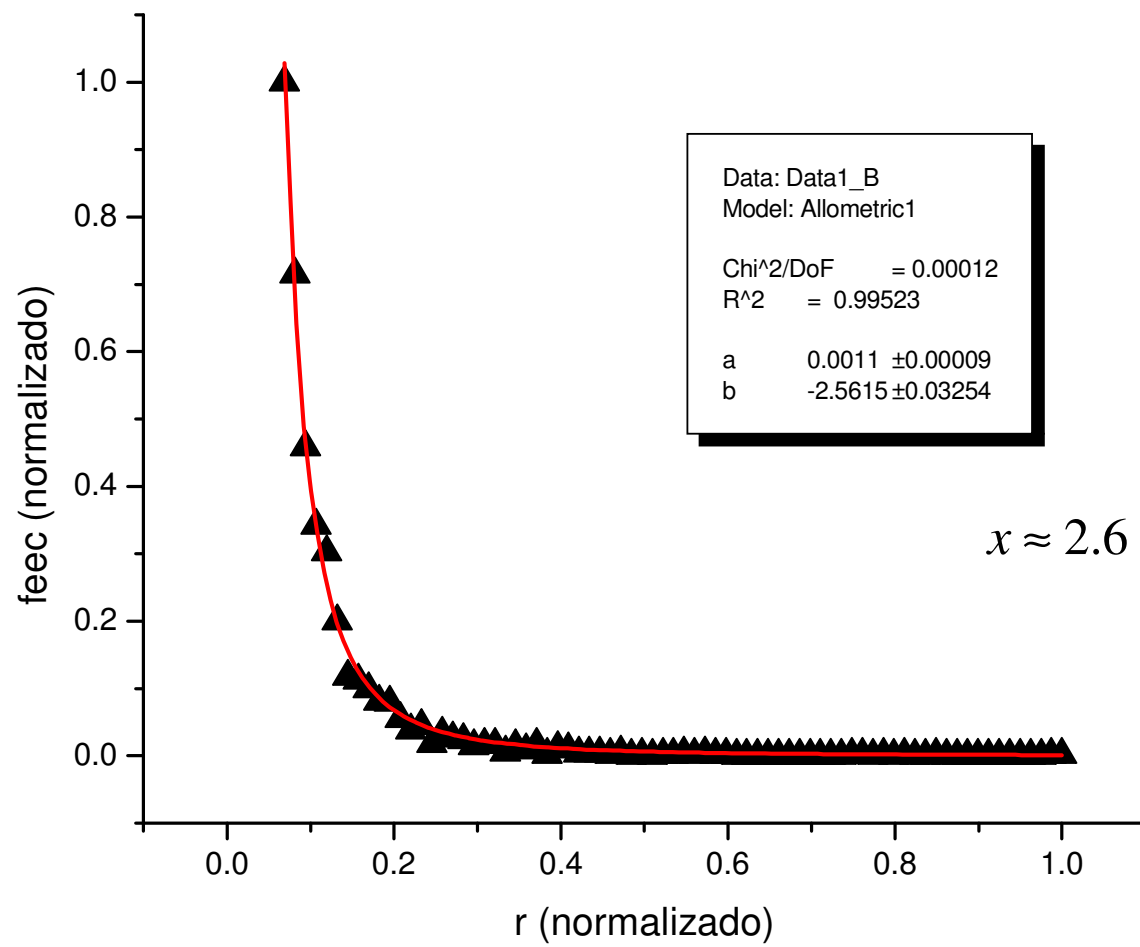
Bethe



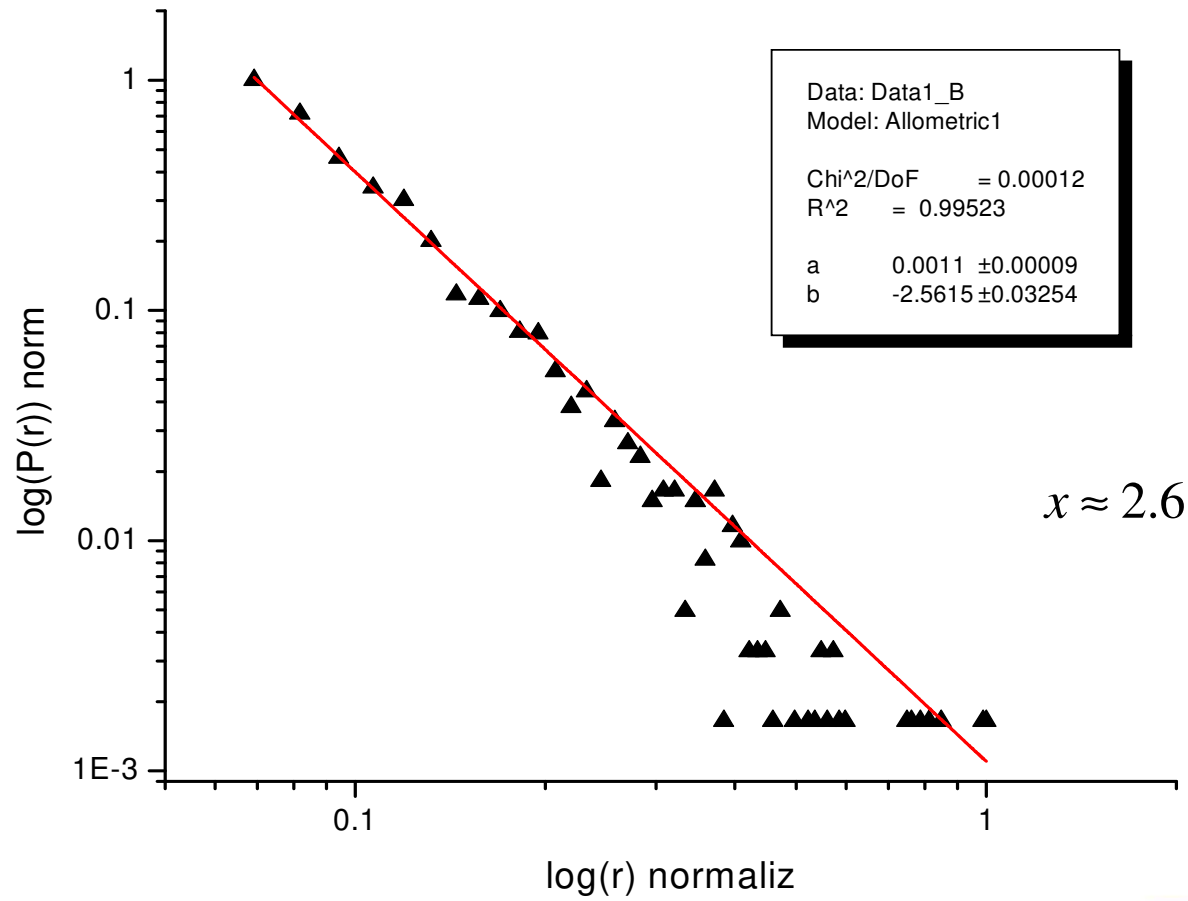
$$p > p_c \Rightarrow f(r) \sim r^{-x} \quad \text{No characteristic scale}$$

We tested this result (Levy distribution?)





log-log:



Well, is scaling, asymptotics of Levy!



$$Q(R) = \int_0^R f(r)Q(r)dr \quad \text{as} \quad f(r) \sim r^{-x}$$

$$Q(r) \sim r^4$$

$$R^\Delta = R_1^\Delta + R_2^\Delta$$

Direct measure of branches (Capulin)

$$\Delta_{\text{exp}} \in (1.82, 1.96) \quad \Delta \approx 1.9$$

Direct measure of vessels: $x \approx 2.6 \pm 0.03 \Rightarrow x \in (2.53, 2.59)$

$$\Rightarrow \Delta_{\text{vasos}} \in (2.41, 2.47)$$

Error in 0.45 (18%)!!



CONCLUSIONS

- 1- resistance to wind stress and nurture are both important criteria in Leonardo's rule
- 2- vessels are Levy distributed, though fitting of critical exponent with experiment is not as good as we expected.
- 3.- Bethe lattice model must be improved to obtain a better fitted critical exponent

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